On a Generalization of the Gronwall–Bellman Lemma in Partially Ordered Banach Spaces

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Submitted by R. Bellman

1. Introduction

The purpose of this paper is to review and unify certain types of functional inequalities which claim their origin to the following

**Lemma** [Gronwall [13], 1919]. Let \( u(t) \) and \( g(t) \) be nonnegative, continuous functions on \( 0 \leq t \leq \tau \), for which the inequality

\[
    u(t) \leq \eta + \int_0^t g(s) u(s) \, ds \quad (0 \leq t \leq \tau)
\]

(1)

holds, where \( \eta \) is a nonnegative constant. Then

\[
    u(t) \leq \eta \exp \left( \int_0^t g(s) \, ds \right) \quad (0 \leq t \leq \tau).
\]

(2)

The above lemma, which provides bounds on solutions of (1) in terms of the solution of a related linear integral equation

\[
    v(t) = \eta + \int_0^t g(s) v(s) \, ds
\]

(3)

* Research supported by The Department of Army under Project No. IT 061101A91A. Present address: U. S. Army Research Office, Durham, North Carolina 27706.

† Research supported by Contract AF 49 (638)-1661 between the office of Scientific Research of the U. S. Air Force and Rensselaer Polytechnic Institute.
is one of the basic tools in the theory of differential equations. On the basis of various motivations it has been extended and used considerably in various contexts. For instance, in the Picard–Cauchy type of iteration for establishing existence and uniqueness of solutions, this lemma and its variants play a significant role (for an extensive bibliography in this connection see Walter [32]). Inequalities of this type (1) are also encountered frequently in the perturbation and stability theory of differential equations (see, for instance, Bellman [3]). For the sake of brevity and in view of the repeated mention of the above lemma in the remainder of the paper, we shall refer to it as the "fundamental lemma" (f.l.).

Section 2 presents a brief survey of linear generalizations of the f.l. Some discrete analogues are discussed in Section 3. Section 4 is devoted to several nonlinear extensions of the f.l. By way of unification, Section 5 deals with inequalities involving operators acting in partially ordered linear spaces. As an application of the results in Section 5, and because of their great practical significance (for instance in the theory of optimal control [4] and scattering processes [6]), matrix inequalities analogous to the f.l. are discussed in the last section. Theorems labelled with letters (A through F) are known; Theorem 1 in Section 3, Theorems 2 and 3 in Section 5, and Theorem 4 in Section 6 appear to be new. References preceded by an asterisk (*) are not mentioned in the text.

## 2. LINEAR GENERALIZATIONS

Among the early users of the f.l. in the theory of ordinary differential equations was Reid [27], who employed a slightly more general form of the f.l. to study the properties of solutions of infinite systems of ordinary linear differential equations. In this paper it is neither our intention, nor is it feasible, to give a complete and up-to-date account of all extensions of the f.l. We shall, however, indicate some representative generalizations and emphasize their interconnections.

A fairly general linear version of the f.l. may be stated as follows:

**Theorem A** (Chu–Metcalf [10], 1967). *Let the functions $u(t), f(t)$ be continuous on the interval $0 \leq t \leq \tau$; let the function $g(t, s)$ be continuous and nonnegative on the triangle $0 \leq s \leq t \leq \tau$. If*

\[
  u(t) \leq f(t) + \int_0^t g(t, s) u(s) \, ds \quad (0 \leq t \leq \tau),
\]

*then*

\[
  u(t) \leq f(t) + \int_0^t G(t, s) f(s) \, ds \quad (0 \leq t \leq \tau)
\]
where

\[ G(t, s) = \sum_{i=1}^{\infty} g_i(t, s) \quad (0 \leq s \leq t \leq \tau) \]

is the resolvent kernel and \( g_i \) (\( i = 1, 2, \ldots \)) are the iterated kernels of \( g \).

As pointed out by the authors in [10], the cases in which one obtains an explicit bound on \( u \) are precisely those in which the resolvent kernel (or a majorant of it) can be summed in a closed form. This is, in fact, the case when \( g(t, s) = h(t) g(s) \geq 0 \). Of particular interest is the case \( h \equiv 1 \).

**Theorem B** (Jones [17], 1964). Let \( u(t) \), \( f(t) \), and \( g(t) \) be real-valued piecewise-continuous functions defined on a real interval \( 0 \leq t \leq \tau \) and let \( g \) be nonnegative on this interval. If

\[ u(t) \leq f(t) + \int_{0}^{t} g(s) u(s) \, ds \quad (0 \leq t \leq \tau), \]

then

\[ u(t) \leq f(t) + \int_{0}^{t} g(s) f(s) \exp \left( \int_{s}^{t} g(\theta) \, d\theta \right) \, ds \quad (0 \leq t \leq \tau). \]

In addition, several generalizations of Theorem B, including subsequent extensions to discrete and discontinuous functional equations, are contained in [17]. Note that the inequality (5) provides the best possible result in the sense that when we replace the inequality (4) by an equality, the same may be done in (5). Also, it is clear that when \( f(t) \equiv \eta \) (a constant), straightforward integration in (7) yields

\[ u(t) \leq \eta \exp \left( \int_{0}^{t} g(s) \, ds \right) \]

which is precisely (2). An alternate form for (7) with slightly stronger assumptions on \( u \) and \( f \) is presented in the book by Sansone and Conti [p. 11, 29].

Another interesting linear generalization is due to Willet [33] under the assumption that either \( g(t, s) \) or \( (\partial/\partial t) g(t, s) \) is degenerate or directly separable in the following sense:

\[ g(t, s) \leq \sum_{i=1}^{n} h_i(t) g_i(s) \]

or a similar relation holds for \( (\partial/\partial t) g(t, s) \).

Inequalities similar to the f.i. but involving functions of several variables (and originally due to Wendroff) may be found in [p. 154, 2].
3. DISCRETE ANALOGUES

Recurrent inequalities involving sequences of real numbers, which may be considered as discrete analogues of the f.l., have been extensively used in the analysis of finite difference equations. For an elementary introduction to application of such results to numerical solutions of ordinary differential equations we refer to the book by Henrici [15]. Discrete analogues of the f.l. have also proved to be very useful in the numerical solutions of partial differential equations. Before we mention some of the typical results in this direction, we prove the following

**Theorem 1.** Let \( m \) be a positive integer, \( u_0, u_1, \ldots, u_m \) a sequence of \( (m + 1) \) nonnegative numbers, and \( z_0, z_1, \ldots, z_m \) a nondecreasing sequence of \( (m + 1) \) real numbers.

Further, let \( \{f_m\} \) be a nonnegative nondecreasing sequence and \( L \geq 0 \). Suppose

\[
\sum_{j=0}^{l-1} u_j (z_{j+1} - z_j) 
\]

is valid for \( l = 1, 2, \ldots, m \). Then the inequality

\[
u_l \leq \{f_1 + Lu_0(z_1 - z_0)\} \prod_{j=1}^{l-1} [1 + L(z_j - z_{j-1})]
\]

holds for \( l = 1, 2, \ldots, m \).

**Proof.** Set \( h_j = (z_{j+1} - z_j) \), \( j = 0, 1, \ldots, m - 1 \). By hypothesis

\[
u_l \leq f_1 + Lu_0h_0 + L \sum_{j=1}^{l-1} u_j h_j.
\]

Since \( 1 + Lh_0 \geq 1 \), the inequality (10) certainly holds for \( l = 1 \). Suppose it is true for \( l \leq n - 1 \). Then we will show that it is true for \( l = n \).

\[\text{1 The authors wish to express their appreciation to Professor J. B. Diaz for several helpful discussions of this material.}\]
Now

\[ u_n \leq (f_n + Lu_0h_0) + L \sum_{j=1}^{n-1} u_j h_j \]

\[ \leq (f_n + Lu_0h_0) + L \sum_{j=1}^{n-1} h_j (f_j + Lu_0h_0) \prod_{i=1}^{j} (1 + Lh_{i-1}) \]

\[ \leq (f_n + Lu_0h_0) \left\{ 1 + L \sum_{j=1}^{n-1} h_j \prod_{i=1}^{j} (1 + Lh_{i-1}) \right\} \]

\[ \leq (f_n + Lu_0h_0) \prod_{j=1}^{n} (1 + Lh_{j-1}) \]

for \{f_n\} is nondecreasing and

\[ \left\{ 1 + L \sum_{j=1}^{n-1} h_j \prod_{i=1}^{j} (1 + Lh_{i-1}) \right\} \]

\[ = 1 + Lh_1(1 + Lh_0) + Lh_2(1 + Lh_0)(1 + Lh_1) + \cdots + Lh_{n-1}(1 + Lh_{n-2}) \]

\[ \leq (1 + Lh_0) \left\{ 1 + Lh_1 + Lh_2(1 + Lh_1) + \cdots + Lh_{n-1}(1 + Lh_{n-2}) \right\} \]

\[ = (1 + Lh_0)(1 + Lh_1) \cdots (1 + Lh_{n-1}) \]

\[ = \prod_{j=1}^{n} (1 + Lh_{j-1}). \]

This completes the proof.

By setting \( f_i = \epsilon \) in Theorem 1 we arrive at the “convergence inequality” which Diaz [12] employed in developing an analogue of the classical Euler–Cauchy polygon method for the solutions of characteristic boundary value problems for a class of nonlinear hyperbolic equations. Similarly, in the investigation of convergence properties of several finite difference schemes for nonlinear parabolic equations, Lees [21] has used the following

**Theorem C** (Lees [21], 1959). *Let u and f be nonnegative functions defined on the integers 1, 2, ..., m. Let f be nondecreasing. If*

\[ u_l \leq f_l + Lk \sum_{i=1}^{l-1} u_i, \quad (l = 1, 2, \ldots, m) \]

(11)
where $L$ and $k$ are positive constants, then

$$u_i \leq f_i \exp(Lk) \quad (l = 1, 2, \ldots, m). \quad (12)$$

Theorem C is readily derived by setting $u_0 = 0$ and $(z_j - z_{j-1}) = k$, $k > 0$, for $j = 1, 2, \ldots, m$. For, under these assumptions, (10) is

$$u_i \leq f_i \prod_{j=1}^{l} (1 + Lk) \leq f_i \exp(Lk).$$

For other useful inequalities which may be considered as discrete analogues of f.l. (or its variants) we refer to Hull and Luxemburg [16], Jones [17], Li [22], and Willet and Wong [34].

4. Nonlinear Generalization

In this section we review some nonlinear generalizations of the f.l. which include as special cases the results of earlier sections. An explanatory remark is in order at this point. Although the nature of the results in this section remains the same as before, namely, a comparison of solutions of certain inequalities with the solutions of the corresponding equations, the nonlinear case differs from the linear case in that the related (comparison) problem will now, in general, be nonlinear, possessing perhaps more than one solution. It is natural therefore to anticipate similar results, but in terms of extremal (maximal, minimal) solutions of the related equation. For instance, consider

**Theorem D** (Opial [24], 1957). Let the mapping $f : [0, \tau] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and satisfy for any $x, y \in \mathbb{R}^n$

$$x \leq y \Rightarrow f(t, x) \leq f(t, y).$$

(Here the relation "\(\leq\)" between any two points $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ means that

$$x \leq y \iff x_i \leq y_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$ )

If the continuous function $u(t)$ mapping $[0, \tau]$ into $\mathbb{R}^n$ satisfies the inequality

$$u(t) \leq \eta + \int_{0}^{t} f(s, u(s)) \, ds \quad (0 \leq t \leq \tau) \quad (13)$$

where $\eta$ is an $n$-vector in $\mathbb{R}^n$, then

$$u(t) \leq \phi(t) \quad (0 \leq t \leq \tau) \quad (14)$$
where $\phi$ is the maximal solution of

$$x(t) = \eta + \int_0^t f(s, x(s)) \, ds \quad (0 \leq t \leq \tau). \quad (15)$$

Theorem D in the special case of $n = 1$ was first established by Viswanatham [30]. This theorem may easily be modified to include the case when $\eta$ itself is a continuous map of $[0, \tau]$ into $\mathbb{R}^n$ and $f$ depends on three arguments $t, s, u$. (See, for instance, [31, cor. 1]). In view of the preceding remark, it is readily seen that the results of Section 2 may be viewed as special cases of Theorem D. Moreover, one also obtains as a special case ($n = 1$ and $f(t, u) = g(t) \omega(u)$ where $g(t) \geq 0$ and $\omega(u)$ is nondecreasing in $u$) a useful generalization of the f.i. due to Bihari [7]. Other results, concerning bounds on the norms of solutions [19, 20] and comparison of solutions [11], can also be obtained readily as particular cases of [31].

It is significant that in all the above results use is made of the fact that the nonlinear function $f$ is nondecreasing in its second argument. In fact, the above results may not hold if $f$ is nonincreasing instead of nondecreasing. By considering the second iterate of the mapping defined by the right-hand side of (13), however, Ziebur has proved the following result, in which $f$ may be either nondecreasing or nonincreasing.

**THEOREM E** (Ziebur [36], 1967). **Define the operator $P$ by**

$$P x(t) = \eta + \int_0^t f(s, x(s)) \, ds \quad (0 \leq t \leq \tau). \quad (16)$$

**Let** $f(t, u)$ **be continuous and be either nondecreasing or nonincreasing in its second argument. Suppose the integral equation**

$$x(t) = P x(t) \quad (17)$$

**has a maximal solution $\phi(t)$. If a continuous function $u(t)$ satisfies**

$$u(t) \leq P u(t) \quad (0 \leq t \leq \tau), \quad (18)$$

**then**

$$u(t) \leq \phi(t) \quad (0 \leq t \leq \tau). \quad (19)$$

If $f$ is nondecreasing and continuous and if $\phi$ is the maximal solution of (15), it is shown [36] that $\phi$ is also the maximal solution of (17). Further, (18) is satisfied whenever (13) holds. Hence Theorem D is contained in Theorem E.
5. Inequalities in Partially Ordered Spaces

For simplicity we shall restrict our consideration to real Banach spaces. Let $B$ denote a real Banach space and let $K \subseteq B$ be a cone [18] of "positive" elements. A partial ordering may be introduced in $B$ in the following way: for $x, y \in B$, $x \leq y$ iff $(y - x) \in K$.

In spaces of common interest like $C[0, \tau], L_p[0, \tau]$, etc., a natural choice for $K$ is the cone of nonnegative functions. In these cases, the partial ordering assumes a simple meaning. For instance, if $u, v \in C[0, \tau]$, then $u \leq v$ means that $u(t) \leq v(t)$ for all $t \in [0, \tau]$.

Consider the operator equation

$$u = Nu + p$$

(20)

where $p$ is a fixed element in $B$ and $N$ is an operator (in general, nonlinear) mapping $B$ into $B$. Throughout we shall assume that for all $u \in B$ the following holds:

$$Nu + p \leq Mu + q$$

(21)

where $q$ is a fixed element in $B$ and $M$ maps $B$ into $B$. Then clearly any solution of (20) will satisfy the operator inequality

$$u \leq Mu + q.$$  

(22)

(22) may be regarded as an abstract analogue of the inequality (1). In this section we obtain bounds on the solutions of (22) in terms of the solutions of the corresponding equation

$$v = Mv + q.$$  

(23)

The following hypothesis is common to Theorems 2 and 3 below:

For all $u, v \in B$, let $M$ satisfy

$$\| Mu - Mv \| \leq \omega(\| u - v \|)$$

(24)

where $\omega(r)$ is nonnegative and continuous for $r \geq 0$.

**Theorem 2.** Let $\omega(r) < r$ for $r > 0$ and let $M$ or $N$ or both be monotonic. Then the unique solution $\phi$ of (23) is an upper bound on all solutions of (22).
Proof. From the hypothesis on $\omega$ it is clear that $M$ is a nonlinear contraction on $B$. Hence in view of [8] the proof of Theorem 1 in [9] may be modified to complete the proof of the theorem.

**Theorem 3.** Let $M$ be completely continuous and monotonic on $B$ and let $\omega(r)$ be a nondecreasing function of $r$ with the following property:

(A) there exists $r^* > 0$ such that for $r \geq r^*$

$$\omega(r) + \|q\| + \|M\theta\| \leq r.$$ 

If $\phi$ is the maximal solution of (23), then $\phi$ is an upper bound on all solutions of (22).

**Proof.** Let $u$ be any solution of (20). For $n = 1, 2, \ldots$, set

$$v_n = Mv_{n-1} + q, \quad v_0 = u, \quad (25)$$

then

$$v_1 = Mv_0 + q \geq Mu + q \geq u = v_0$$

where we have used (22). From the monotonicity of $M$, an induction on $n$ shows that \{v_n\} is a nondecreasing sequence in $B$. If now we suppose that for some $n$, $\|v_n\| \leq R$, $R > 0$, then (25) in conjunction with (24) and the monotonicity of the scalar function $\omega$ gives

$$\|v_{n+1}\| \leq \omega(\|v_n\|) + \|M\theta\| + \|q\| \leq \omega(R) + \|M\theta\| + \|q\|.$$ 

Therefore, by invoking property (A), we conclude that $\|v_{n+1}\| \leq R$ whenever $R \geq r^*$. Recall that $M$ is a completely continuous operator; hence for the compactness of the sequence \{v_n\}, we need only select $R \geq \max(\|u\|, r^*)$. Thus, the monotonic and compact sequence \{v_n\} converges to some $v \in B$, and the continuity of $M$ implies that $v$ is, in fact, a solution of (23). But from the maximality, $v \leq \phi$. Thus $u \leq v \leq \phi$, which was to be proved.

As an illustration, Theorem D of Section 4 can be shown to be a consequence of Theorem 3. Let $B = C^n[0, \tau]$ be the Banach space of continuous vector functions (with $n$ components) on $0 \leq t \leq \tau$, and $K$ the cone consisting of those functions in $C^n[0, \tau]$ whose components are nonnegative on $0 \leq t \leq \tau$. Define an operator $M$ mapping $C^n[0, \tau]$ into itself as follows: for all $x \in C^n[0, \tau]$

$$Mx = \int_0^t f(s, x(s)) \, ds,$$
where $f$ is the function in Theorem D which is assumed to be continuous and nondecreasing in its second argument. An application of Arzela's Theorem guarantees that $M$ is a completely continuous operator, and the monotonicity of $M$ is a consequence of the monotonicity of $f$. Now for any $u, v \in B$ such that $\| u \| \leq R$, $\| v \| \leq R$, where $0 < R < \infty$, we observe that

$$\| Mu - Mv \| \leq 2\tau F$$

Thus by choosing for $\omega(r)$ the identically constant function $2\tau F$, we can satisfy the remaining hypotheses of Theorem 3.

It may, be remarked, however, that an application of either Theorem 2 or 3, in general, poses the formidable task of choosing properly the scalar function $\omega(r)$. This, in turn, will depend on a judicious selection of a "majorant" operator $M$ for a given problem so that the inequality (21) holds.

Note that Theorems 2 and 3 are valid in any Banach Space, as we have made no assumptions on the spaces involved. We shall mention now some related results in which the structure of the cone $K$ plays a crucial role. There have been many generalizations of the f.i. in this direction. For instance, using a lattice fixed point theorem, Hanson and Waltman [14] obtained such results for functional inequalities. Their results, in particular, include a generalization of the f.i. due to Viswanatham [31]. In a similar context, a typical result for operator inequalities in partially ordered linear spaces is contained in a paper by Pelczar (where additional useful references can be found):

**Theorem F** (Pelczar [25], 1965). *Let $M$ be a monotonic operator mapping $B$ into $B$. Define the subset $Q \subset B$ as follows:

$$Q = \{ z \mid z \in B, z \leq Mz + q \}, \quad q \in B.$$*

If $Q$ is nonempty and $\sup Q$ (say $\bar{v}$) exists, then $\bar{v}$ is the maximal solution of

$$v = Mv + q.$$*

The assumption that $\sup Q$ exists is, however, a restriction on the cone $K$ which may exclude many spaces of practical interest. For instance, it may not be true in $C[0, \tau]$, the space of continuous functions, which is partially ordered by the cone of nonnegative functions in $C[0, \tau]$ (see [p. 50, 18]).
6. Matrix Inequalities

Let \( A \) denote the linear space of real \( n \times n \) symmetric matrices. In \( A \), one can introduce a partial ordering in more than one way. For instance, using, respectively, cones of nonnegative matrices and nonnegative definite matrices, two different types of orderings can be introduced in \( A \). Since a nonnegative definite matrix is a natural generalization of a nonnegative number, we adopt the second kind of ordering (that is, the one induced by the cone \( \mathcal{H} \) of nonnegative definite matrices). Then with this ordering in \( A \) we have

\[
X, Y \in A, \quad X \preceq Y \quad \text{iff} \quad (Y - X) \in \mathcal{H}.
\]

A function \( P : A \rightarrow A \) is called monotonic [2] if \( X, Y \in A \) and \( X \preceq Y \) imply \( P(X) \preceq P(Y) \) (that is, \( P(Y) - P(X) \) is a nonnegative definite matrix).

**Theorem 4.** Let \( H \) be a real symmetric matrix. Let \( G \) be a monotone and Lipschitz continuous function from \( A \) into \( A \):

\[
||G(X) - G(Y)|| \leq \rho \|X - Y\|.
\]  
(26)

Then the inequality 

\[
X(t) \preceq H(t) + \int_0^t G(X(s)) \, ds
\]

implies 

\[
X(t) \preceq Y(t)
\]

on their common interval of existence, where \( Y(t) \) is the unique solution of the corresponding equality.

**Proof.** For \( n = 1, 2, \ldots, \) set 

\[
Y_n(t) = H(t) + \int_0^t G(Y_{n-1}(s)) \, ds
\]

where \( Y_0(t) = X(t) \in A \). Then \( \{Y_n\}, n = 1, 2, \ldots, \) are all in \( A \). Next, using the monotonicity of \( G \), it is easily verified that 

\[
X(t) \preceq Y_1(t) \preceq \cdots \preceq Y_n(t).
\]

Since \( G \) is Lipschitz continuous, \( \{Y_n(t)\} \) converges to the unique solution \( Y(t) \) of the corresponding equality. This completes the proof.

The following results used by Bellman [5] may be regarded as corollaries of the above theorem.
Corollary 1.

\[ \frac{dX}{dt} \leq F(t) + G(X), \quad X(0) = C \]

implies

\[ X(t) \leq Y(t) \]

where \( F \) and \( C \) are real and symmetric, \( G \) has the properties above, and \( Y(t) \) is the unique solution of the initial value problem

\[ \frac{dY}{dt} = F(t) + G(Y), \quad Y(0) = C. \]

Proof. Integrating the inequality gives

\[ X(t) \leq C + \int_0^t F(s) \, ds + \int_0^t G(X(s)) \, ds. \]

If we set

\[ H(t) = C + \int_0^t F(s) \, ds \in \mathcal{A}, \]

the result follows immediately from Theorem 4.

Corollary 2.

\[ \frac{dX}{dt} \leq F(t) + RX + XRT + \sum_{i=1}^m Q_i X Q_i^T, \quad X(0) = C \]

implies

\[ X(t) \leq Y(t) \]

where \( F \) and \( C \) are real and symmetric, \( R \) and \( Q_i \) \((i = 1, 2, \ldots, m)\) are real constant \( n \times n \) matrices, and \( Y(t) \) is the unique solution of the corresponding initial value problem.

Proof. Following a familiar procedure (multiplying from the left and right by \( e^{Rt} \) and \( e^{R^Tt} \), respectively, etc.)

\[ X(t) \leq e^{Rt}Ce^{R^Tt} + \int_0^t e^{R(t-s)} \left[ F(s) + \sum_{i=1}^m Q_i X Q_i^T \right] e^{R^T(t-s)} \, ds \]

\[ - H(t) + \int_0^t G(t, s, X(s)) \, ds \]
where we have set

\[ H(t) = e^{Rt}Ce^{RTt} + \int_0^t e^{R(t-s)}F(s)e^{RT(t-s)}\, ds \]

and

\[ G(t, s, x) = e^{R(t-s)} \sum_{i=1}^m Q_iXQ_i^Te^{RT(t-s)}. \]

Clearly \( H(t) \in \mathcal{A} \). Also \( G \) is Lipschitz continuous in its last argument and monotonic, because if \( X \leq Y \), for each \( i \quad (i = 1, 2, \ldots, m) \)

\[ Q_iYQ_i^T - Q_iXQ_i^T = Q_i(Y - X)Q_i^T \in \mathcal{X} \]

for any \( Q_i \). Again,

\[ e^{R(t-s)} \left[ \sum_{i=1}^m Q_i(Y - X)Q_i^T \right] e^{RT(t-s)} \in \mathcal{X}. \]

Having established the desired properties for \( H \) and \( G \), we may now apply Theorem 4 to obtain our conclusion.

REFERENCES

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If is a partial metric space and is partially ordered, then is called an ordered partial metric space. Then, are called comparable if or holds. Let be two self-mappings, is said to be -nondecreasing if implies for all . If is the identity mapping on , then is nondecreasing. Starting from the concept of partially ordered set, the existence of fixed points in ordered metric spaces was largely investigated by many researchers, some of these are Turinici, Ran and Reurings, Nieto and Rodríguez-López. For more details on this topic, we also refer to , , , and references therein. G.E. Hardy and T.D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull. Application of fixed point results in partially ordered metric spaces was made subsequently, for example, by Ran and Reurings [18] to solving matrix equations and by Nieto and Rodriguez-Lopez [19] to obtain solutions of certain partial differential equations with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point results in partially ordered metric spaces, some of which are in2021222324252627282930313233. The purpose of this paper is to establish some fixed point results satisfying a generalized contraction mapping of rational type in met Banach [7] contraction principle is a powerful tool for solving many problems in applied mathematics and sciences. It has been improved and extended in many ways. Most of the fixed point theorems in nonlinear analysis usually start with Banach [7] contraction principle. Mustafa [22] gave a generalization of Banach contraction principle in complete ordered partial b - metric space. The notion of a coupled fixed point was introduced and studied by Opoitsev [25] and then by Guo and Lakshmikantham [12]. In fact, we show that coupled fixed point theorems on a complete partially ordered partial b - metric. space with coefficient of partial b - metric space s â” 1 by, using monotone property under Geraghty - type contraction.